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STABILITY OF RELATIVE EQUILIBRIUM OF A BODY IN A PERTURBED CIRCULAR ORBIT

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We study the stability of relative equilibrium of a body whose center of mass describes a non-Keplerian circular orbit without a center of attraction, under the action of perturbing or controlling forces. The problem is solved under a restricted formulation, in which only the gravitational moments relative to the central field are taken into account. Sufficient conditions of stability of the positions of equilibrium obtained are found, using the Routh theorem in the manner analogous to that developed in [1].

The problem of relative equilibrium of a rigid body in a circular unperturbed orbit and its stability, were investigated recently in detail by many authors [1, 2]. We find however, that in certain concrete problems of celestial mechanics and dynamics of space flights there is a need to generalize this problem to the case of perturbed circular orbits which are realized when perturbing or controlling forces are present. An example of such a problem is the case of circular non-Keplerian orbits in the gravitational field of an axisymmetric planet. The author of [3] has proved, in particular, the existence of circular orbits the plane of which is parallel to the equatorial plane of the planet.

An interesting problem from the point of view of space dynamics is that of setting a synchronous stationary satellite (SS) at an arbitrary latitude. The authors of [4, 5] studied this problem for the center of mass of the satellite, and [6] dealt with the translational-rotational motion of such a satellite under the assumption that an additional constant reactive force is applied to its center of mass. It was shown that a relative equilibrium of a body is possible when its center of mass is in circular motions in a plane that does not contain the center of attraction. This problem was not previously investigated.

The aim of the present paper is to obtain sufficient conditions for the relative equilibrium of a rigid body when the formulation of the problem is restricted, i.e. under the assumption that the motion of the body relative to the center of mass does not influence the motion of the center of mass itself. We assume, moreover, that the motion of the body relative to the center of mass is induced only by the gravitational moment of the central Newtonian field.

Let φ and r denote, respectively, the latitude and the distance of the center of mass of a body moving along a circular orbit from the center of attraction O, and we attach to this center a stationary rectangular $O\xi\eta\zeta$ coordinate system the axis $O\zeta$ of which is perpendicular to the plane of the orbit. At the center of mass C we place the origins of the two rectangular coordinate systems: the axes of the first system $(Cxyz)$ which we shall call orbital coordinate system, will be oriented in the direction of velocity of the center of mass, tangentially to the meridian, and along the radius vector of the center of mass, while the axes of the other $(Cx'y'z')$ system will coincide with the principal axes of inertia of the body.

Using the expansion of the attraction force function for a rigid body in a central Newtonian field [1], we set the following expression for the altered potential energy of the body:

$$W_* = \frac{3}{2}\frac{k}{r^3}(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2) - \frac{\omega^2}{2}(A\beta_1^2 + B\beta_2^2 + C\beta_3^2)$$

Here A, B and C are the moments of inertia relative to the Cx', Cy' and Cz' axes, respectively, $\gamma_1, \gamma_2, \gamma_3$ and $\beta_1, \beta_2, \beta_3$ are the direction cosines of the Cz and $O\zeta$ axes relative to the same (associated) axes, ω is the angular velocity of rotation of the center of mass along the orbit and k is the gravitational parameter of the central field.

Since we are planning to use the Routh theorem to show the existence of the positions of relative equilibrium of the body and to investigate their stability, we shall make use of the method of solving such problems developed by Rumiantsev in [2]. Let us therefore eliminate the cosines β_2 and γ_3 using the geometrical relations, and write W_* in the form

$$W_* = \frac{\omega^2}{2}[3\mu^2(A-C)\gamma_1^2 + 3\mu^2(B-C)\gamma_2^2 + (B-C)\beta_3^2 + (B-A)\beta_1^2 - B] \quad (1)$$

where $\mu^2 = \omega^2 r^3 / k$. Using another obvious geometrical relation

$$\chi \equiv \beta_1\gamma_1 + (1-\beta_1^2-\beta_3^2)^{1/2}\gamma_2 + \beta_3(1-\gamma_1^2-\gamma_2^2)^{1/2} - \sin\varphi = 0$$

we shall seek the extremum of the function

$$W = 2\omega^{-2}W_* + \lambda\chi$$

where λ is the Legendre multiplier. Then the positions of relative equilibrium will correspond to the solutions of the equations

$$\frac{\partial W}{\partial\gamma_1} = 6\mu^2(A-C)\gamma_1 + \lambda\left[\beta_1 - \frac{\gamma_1\beta_3}{(1-\gamma_1^2-\gamma_2^2)^{1/2}}\right] = 0$$

$$\frac{\partial W}{\partial\beta_1} = 2(B-A)\beta_1 + \lambda\left[\gamma_1 - \frac{\beta_1\gamma_2}{(1-\beta_1^2-\beta_3^2)^{1/2}}\right] = 0$$

$$\frac{\partial W}{\partial\gamma_2} = 6\mu^2(B-C)\gamma_2 + \lambda\left[(1-\beta_1^2-\beta_3^2)^{1/2} - \frac{\gamma_2\beta_3}{(1-\gamma_1^2-\gamma_2^2)^{1/2}}\right] = 0$$

$$\frac{\partial W}{\partial\beta_3} = 2(B-C)\beta_3 + \lambda\left[(1-\gamma_1^2-\gamma_2^2)^{1/2} - \frac{\gamma_2\beta_3}{(1-\beta_1^2-\beta_3^2)^{1/2}}\right] = 0$$

which give the conditional extremum of the function (1). The solution has the form

$$\gamma_1 = \beta_1 = 0, \quad \gamma_2 = \sin\alpha, \quad \beta_3 = \sin(\varphi-\alpha)$$
$$\lambda = (C-B)\frac{\sin 2(\varphi-\alpha)}{\cos\varphi}, \quad \operatorname{tg} 2\alpha = \frac{\sin 2\varphi}{3\mu^2 + \cos 2\varphi} \quad (2)$$

and corresponds to the position of relative equilibrium of the body in which one of its principal axes of inertia (Cx') coincides with the velocity of the center of mass, and the remaining two axes form the angle α with the axes of the orbital system, the angle defined by the last formula of (2).

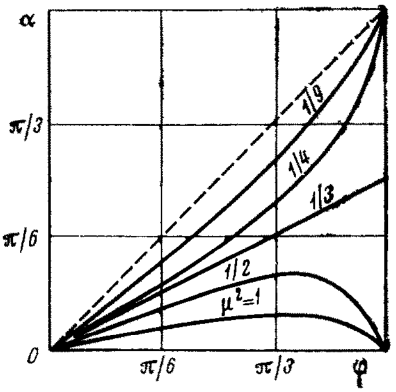

Fig. 1

When $\varphi = 0$, the position of equilibrium just obtained becomes position of equilibrium of a rigid body in a circular Keplerian orbit irrespective of the value of μ. It is interesting to note that this two-parameter family of relative equilibria is independent of the geometry of the body mass and of the orbit radius, and is fully defined by the latitude φ and parameter μ. Figure 1 shows the relationship $\alpha(\varphi)$ for various values of μ.

In order to investigate the stability of the solutions obtained we write the conditions of positive definiteness of the function W and this, together with the Routh theorem, yields the sufficient conditions of stability.

Constructing the second order derivatives of W and taking into account (2), we use the Silvester criterion to obtain the following conditions of positive definiteness of the function W:

$$(B-C)(1-\Phi) > 0, \qquad A - C + (C-B)\Phi > 0 \tag{3}$$

$$[C - A + (B-C)\Phi][(B-C)\Phi + A - B] + \frac{3\mu^2\sin^2\varphi}{p}(B-C)^2 > 0$$

$$(B-C)^2 F(\mu^2, \varphi) > 0$$

where

$$p = 9\mu^4 + 6\mu^2\cos 2\varphi + 1, \qquad \Phi = \frac{1}{2}\left(1 - \frac{1+3\mu^2}{\sqrt{p}}\right)$$

$$F(\mu^2, \varphi) = \frac{6(1+3\mu^2)^2\mu^2}{1 + 6\mu^2\cos^2\varphi + 9\mu^4 + \sqrt{p}\,(1+3\mu^2)}$$

We see that when $0 \leqslant \varphi \leqslant \pi/2$, we have $1 - \Phi > 0$. Consequently the first condition of (3) yields $B > C$. The third inequality of (3), after elementary computations, reduces unexpectedly to

$$(C-A)(A-B) > 0$$

and this, together with the previously obtained inequality $B > C$, yields the Beletskii condition [2]

$$B > A > C \tag{4}$$

We shall show that the second condition of (3) holds for all μ and φ, $(0 < \mu < \infty,\ 0 \leqslant \varphi \leqslant \pi/2)$ provided that (4) holds. Let us therefore write this inequality in the form

$$\varkappa(B-A) > C - A, \qquad \varkappa = \frac{1 + 3\mu^2 - \sqrt{p}}{1 + 3\mu^2 + \sqrt{p}} \tag{5}$$

It can easily be shown that $0 \leqslant \varkappa \leqslant 1$ for all μ and φ within the stated interval, consequently (5) holds whenever (4) holds. When $B \neq C$, the last inequality of (3) yields

$$F(\mu^2, \varphi) > 0$$

which clearly holds over the whole interval of variation of μ and φ.

Thus, the above investigation shows that the position of relative equilibrium in question can be stable at all latitudes φ, while the sufficient conditions of stability fully coincide with the Beletskii condition (4) and do not contain any parameters of the orbit of the center of mass.

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ON THE MAXIMIZATION OF THE DEGREE OF STABILITY OF A LINEAR OSCILLATING SYSTEM

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The problem of selecting optimal parameters ensuring the maximum degree of stability, is considered for the linear oscillating systems [1]. The upper bounds of the degree of stability are obtained. Necessary and sufficient conditions of attainability of the upper bound are formulated. Systems with one, two and three degrees of freedom are studied in detail. Similar problems have been already investigated in [1 – 4].

1. Statement of the problem. We consider a system the motion of which is described by the following linear differential equation:

$$Ax^{\cdot\cdot} + Bx^{\cdot} + Cx = 0 \tag{1.1}$$

Here x is an n-dimensional vector, A, B and C are $n \times n$ matrices and a dot denotes the derivative with respect to time. Equation (1.1) can describe e.g. small oscillations of a mechanical system about the position of equilibrium $x = 0$. Problems of the stabi-